NON-COMMUTATIVE GEOMETRY AND CHIRAL PERTURBATION LAGRANGIAN

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Abstract

Chiral perturbation lagrangian in the framework of non-commutative geometry is considered in full detail. It is found that the explicit symmetry breaking terms appear and some relations between the coupling constants of the theory come out naturally. The WZW term also turns up on the same footing as the other terms of the chiral lagrangian.

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1 Introduction

It has been shown by Connes[1, 2] that the lagrangian of the Standard Model with the Spontaneous symmetry breaking mechanism can be obtained naturally by using the non-commutative geometrical concepts for differentiation, integration, connection, curvature, etc. The Higgs Field appears in non-commutative lagrangian as a generalized gauge field in the direction of discrete dimension [1-6]. In Ref[7] it was shown that the explicit symmetry breaking mechanism may also be related to the existence of an additional discrete dimension. The mass terms in the lagrangian of the Chiral Perturbation Theory (ChPL) were obtained in the framework of Non-Commutative Geometry (NCG) up to the 4th order of momentum. There are two remarkable points in this approach. First, the complicated terms in the ChPL are obtained by simple assumptions and direct calculations; second, some relations between the coupling constants of the theory come out naturally. As the lagrangian of Ref[7] was limited to the 4 dimensional space in the continuous part of the geometrical space, the WZW term was absent. This term can be obtained if we take a 5 dimensional manifold. In Ref[7] also, instead of the field χ which appear in Gasser-Leutwyler lagrangian[8], only the m^2 term (m is the meson mass) appeared. In this paper, by a suitable definition of χ in terms of the mass matrice M and the mesonic field on the other layer of space, we regenerate the original ChPL.

This paper is divided in to four sections as follows. In section 2 we introduce the tools which are needed in NCG. In section 3 we will construct the lagrangian of ChPT in the framework of NCG, and in the last section we will study the WZW term, in this framework.

2 A Brief Review of NCG

There is a well-known theorem due to Gelfand and Naimark that describes how one may substitute a compact topological space $\mathcal{M}$ with the algebra $\mathcal{C}^\infty(\mathcal{M})$ of complex continuous functions defined on $\mathcal{M}$, which has sup norm and is commutative. The extension of

this theorem is the Connes' proposal which considers how to define a compact, non-commutative space in terms of a unital, non-commutative $*$-algebra $\mathcal{A}$[1].

The starting point of Connes approach is the K-cycle $(\mathcal{H}, D)$, where $\mathcal{H}$ is the Hilbert space for representing the elements of $\mathcal{A}$ as the linear operators and D is called the generalized Dirac operator. The role of K- cycle for NCG is similar to the role of differential structure in ordinary differential geometry. Having a K-cycle one can develop a differential algebra $\Omega(\mathcal{A})$ for the non-commutative space which is equivalent to differential geometry for the manifolds. For this purpose assume d to be an abstract differential operator which acts on elements of $\mathcal{A}$ and satisfies the Leibniz rule, with $d1 = 0, d^2 a = 0$ where $1, a \in \mathcal{A}$. Connes defines the p-forms in $\Omega(\mathcal{A})$ as:

$$\alpha = \sum_i a_0^i \, da_1^i \, ... \, da_p^i \quad ; a_0^i, a_1^i, ..., a_p^i \in \mathcal{A}, \tag{1}$$

and also the representation of α in $\mathcal{H}$ is shown by $\pi(\alpha)$ and is defined as:

$$\pi(\alpha) = \sum_i \pi(a_i^0)[D, \pi(a_i^1)]...[D, \pi(a_i^p)]. \tag{2}$$

As a simple example, consider a Euclidean spin manifold $\mathcal{M}$. For such manifold, we should take $\mathcal{A}$ the algebra of complex valued functions on $\mathcal{M}$,

$$\mathcal{A} := \mathcal{C}^\infty(\mathcal{M}), \tag{3}$$

then D is the ordinary Dirac operator [2]

$$D = \not{\partial} = \gamma^\mu \partial_\mu. \tag{4}$$

According to the above definition, we have:

$$\pi(dg) = [D, g], \quad \forall g \in \mathcal{A} \tag{5}$$

which gives

$$\pi(dg) = [\not{\partial}, g] = \gamma^\mu \, \partial_\mu g \equiv \gamma(\mathbf{d}g), \tag{6}$$

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[2]We ignore the i's in our calculations because we can finally absorb them in the coupling constants of the theory.

where $\mathbf{d}g$ is the ordinary one-form, $\mathbf{d}g = \partial_\mu g \mathbf{d}x^\mu$. To describe a fibre bundle over the manifold $\mathcal{M}$, we may choose

$$\mathcal{A} = \mathcal{C}^\infty(\mathcal{M}) \otimes M_N(\mathbb{C}), \tag{7}$$

where $M_N(\mathbb{C})$ is the algebra of $N \times N$ complex matrices. Also one may assume the discrete dimension for geometrical spaces. For instance in the case of a two layer space, each layer is described by the algebra of continuous functions $\mathcal{C}^\infty(\mathcal{M})$. The proper algebra for this geometry is:

$$\mathcal{A} = \mathcal{C}^\infty(\mathcal{M}) \otimes (\ M_N(\mathbb{C}) \oplus M_N(\mathbb{C})\ ). \tag{8}$$

We may take $\mathcal{M}$ as a 4-dimensional Euclidean compact manifold. To make differential algebra, the Dirac operator can be chosen as follows.

$$D = \begin{pmatrix} \partial\!\!\!/ \ \otimes \ \mathbb{1} & \gamma_5 \otimes M \\ \gamma_5 \otimes M^\dagger & \partial\!\!\!/ \ \otimes \ \mathbb{1} \end{pmatrix} \tag{9}$$

where M is an $N \times N$ matrix. A representation of any element $g \in \mathcal{A}$ in $\mathcal{H}$ can be taken as:

$$\pi(g) \equiv \begin{pmatrix} V(x) & 0 \\ 0 & V'(x) \end{pmatrix} \ , \ g \ \in \mathcal{A} \tag{10}$$

where $V(x)$ and $V'(x)$ are $N \times N$ matrices and x indicates the coordinates on $\mathcal{M}$. This representation contains in itself the information about the two layers of space. $V(x)$ represents the functions on one layer and $V'(x)$ those of the other. Now it can easily be shown that:

$$\pi(dg) \ = \ [\,D\,, \pi(g)\,] = \begin{pmatrix} \partial\!\!\!/ V & \gamma_5(M\,V' - V\,M) \\ \gamma_5(M^\dagger V - V' M^\dagger) & \partial\!\!\!/ V' \end{pmatrix}. \tag{11}$$

At the end it would be useful to introduce the unitary groups of the algebra $\mathcal{A}$ as:

$$\mathcal{U}(\mathcal{A}) = \{g \in \mathcal{A} \ , gg^* = g^*g = 1\}, \tag{12}$$

where $*$ indicates the involution in $\mathcal{A}$.

3   Chiral Perturbation Theory

Gasser and Leutwyler in their method of obtaining the ChPL which is the effective la-grangian for low energy QCD, stated that this lagrangian should preserve all the symmetries of QCD. Their method however was not based on using the QCD lagrangian, instead it was completely phenomenological. Their basic idea was expanding the effective lagrangian in terms of different powers of the mesonic field momentum i.e. in terms of different powers of the field derivative[8, 9]. Here we use a similar idea but we utilize the generalized derivative of non-commutative geometry for the mesonic field instead of the ordinary one. We start from a simple minded form of the lagrangian in terms of different powers of $L = L_\mu \mathbf{d}x^\mu = (U\partial_\mu U^\dagger)\mathbf{d}x^\mu$, where $U \in SU(3)_{flavour}$ is the mesonic field (actually $U = e^{\frac{i\lambda^a \pi_a}{2}}$ where π is the pionic field). As we shall see, in our approach all the explicit symmetry breaking terms (mass included terms) appears naturally. More than that we obtain some extra relations between the coupling constants of the theory.

At the beginning we take the following algebra:

$$\mathcal{A} = (\mathcal{C}^\infty(\mathcal{M}) \otimes M_N(\mathbb{C})) \oplus M_N(\mathbb{C}) \tag{13}$$

which describes the geometrical space containing a 4-dimensional manifold and a point which is separated from $\mathcal{M}$ by the additional discrete dimension. The Dirac operator is taken to be (See also Ref[7]):

$$D = \begin{pmatrix} \not{\partial} \otimes \mathbb{1} & \gamma^5 \otimes M \\ \gamma^5 \otimes M^\dagger & 0 \end{pmatrix} \tag{14}$$

where M and $\mathbb{1}$ are 3×3 unitary matrices, and

$$MM^\dagger = M^\dagger M = m^2 \mathbb{1}. \tag{15}$$

If g is an element in $\mathcal{U}(\mathcal{A})$, the unitary group of $\mathcal{A}$, then we generalize the one-form $L = (U\partial_\mu U^\dagger)\mathbf{d}x^\mu$ as follow:

$$L = gdg^* \ , g \in \mathcal{U}(\mathcal{A}). \tag{16}$$

The representation of g in the Hilbert space $\mathcal{A}$ can be taken as:

$$g = \begin{pmatrix} U(x) & 0 \\ 0 & U' \end{pmatrix} \quad , \quad U, U' \in U(N) \tag{17}$$

Then for the representation of L in $\mathcal{H}$ we have:

$$\pi(L) = g[\, D\, , g^*\,] = \begin{pmatrix} U\not\partial U^\dagger & \gamma_5 U(M\, U'^\dagger - U^\dagger M) \\ \gamma_5 U'(M^\dagger\, U^\dagger - U'^\dagger\, M^\dagger) & 0 \end{pmatrix}. \tag{18}$$

To obtain the effective chiral perturbation lagrangian, we first expand the lagrangian in terms of derivatives of the field g or powers of L.

$$\mathcal{L}_{eff} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \dots\ , \tag{19}$$

where $\mathcal{L}^{(n)}$ are the set of terms which contain n derivatives of the field g. $\mathcal{L}^{(0)}$ should be set equal to zero, because according to PCAC all the interactions are momentum dependent[9]. In this direction $\mathcal{L}^{(1)}$ through $\mathcal{L}^{(4)}$ will assume to have the following form:

$$\mathcal{L}^{(1)} = Tr(K_1^{(1)} L) \tag{20}$$

$$\mathcal{L}^{(2)} = Tr(K_1^{(2)} L^2) + [Tr(K_2^{(2)} L)]^2 \tag{21}$$

$$\mathcal{L}^{(3)} = Tr(K_1^{(3)} L^3) + Tr(K_2^{(3)} L)Tr(L^2) + [Tr(K_3^{(3)} L)]^3 \tag{22}$$

$$\mathcal{L}^{(4)} = Tr(K_1^{(4)} L^4) + [Tr(K_2^{(4)} L^2)]^2 + Tr(K_3^{(4)} L)Tr(L^3) \tag{23}$$

$$+ \ Tr(K_4^{(4)} L)[Tr(L)]^3 + [Tr(K_5^{(4)} L)]^4 + [Tr(K_6^{(4)} L)]^2 Tr(L^2)\ , \tag{24}$$

where $K_j^{(i)}$ are some diagonal 2×2 matrices whose elements $(k_j^{(i)}, k_j'^{(i)})$ are the coupling constants of the theory[3]. It is interesting to note that the $\mathcal{L}^{(odd)}$s vanish due to the vanishing of the trace of an odd number of Dirac marices. Now if we take $\chi = MU'M^\dagger$ and directly calculate $\mathcal{L}^{(2)}$ and $\mathcal{L}^{(4)}$ and using the trace identity in ref[8], we will obtain the effective lagrangian up to 4th order of momentum as follows:

$$\mathcal{L} = [-4k_1^{(2)} - 8m^2(3k_1^{(4)} + k_1'^{(4)}) - 192m^2 k_2^{(4)}(k_2^{(4)} + k_2'^{(4)})]\ Tr(\partial_\mu U \partial^\mu U^\dagger)$$

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[3]One may assume $k_j^{(i)}$ is the coupling constants on one layer and $k_j'^{(i)}$ is due to the other (infact the disjoint point).

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$$+ \quad [-4(k_1^{(2)} + k_1'^{(2)}) - 16m^2(k_1^{(4)} + k_1'^{(4)}) - 192m^2(k_2^{(4)} + k_2'^{(4)})^2] \, Tr(\chi U^\dagger + U\chi^\dagger)$$

$$+ \quad (-2k_1^{(4)} + 16(k_2^{(4)})^2) \, [Tr(\partial_\mu U \partial^\mu U^\dagger)]^2 - 4k_1^{(4)} \, Tr(\partial^\mu U \partial^\nu U^\dagger) \, Tr(\partial_\mu U \partial_\nu U^\dagger)$$

$$+ \quad 16k_1^{(4)} Tr(\partial_\mu U \partial^\mu U^\dagger \partial_\nu U \partial^\nu U^\dagger) + 32k_2^{(4)}(k_2^{(4)} + k_2'^{(4)}) \, Tr(\partial_\mu U \partial^\mu U^\dagger) \, Tr(\chi U^\dagger + U\chi^\dagger)$$

$$+ \quad 4(3k_1^{(4)} + k_1'^{(4)}) \, Tr(\partial_\mu U \partial^\mu U^\dagger(\chi U^\dagger + U\chi^\dagger)) + 16(k_2^{(4)} + k_2'^{(4)})^2 \, [Tr(\chi U^\dagger + U\chi^\dagger)]^2$$

$$+ \quad 4(k_1^{(4)} + k_1'^{(4)}) \, Tr(\chi U^\dagger \chi U^\dagger + \chi^\dagger U \chi^\dagger U) + 8(k_1^{(4)} + k_1'^{(4)}) \, Tr(\chi\chi^\dagger) \tag{25}$$

which is the Gasser-Leutwyler Lagrangian[8] except for the external fields . Here we concentrate on the symmetry breaking term which are the χ dependent terms and ignore the external fields[8, 9]. By comparing (25) with the original ChPL[8], one can easily find the following relations:

$$\frac{1}{4}F_0^2 \quad = \quad -4k_1^{(2)} - 8m^2(3k_1^{(4)} + k_1'^{(4)}) - 192m^2 k_2^{(4)}(k_2^{(4)} + k_2'^{(4)}) \tag{26}$$

$$\frac{1}{4}F_0^2 \quad = \quad -4(k_1^{(2)} + k_1'^{(2)}) - 16m^2(k_1^{(4)} + k_1'^{(4)}) - 192m^2(k_2^{(4)} + k_2'^{(4)})^2 \tag{27}$$

$$L_1 \quad = \quad -2k_1^{(4)} + 16(k_2^{(4)})^2 \tag{28}$$

$$L_2 \quad = \quad -4k_1^{(4)} \tag{29}$$

$$L_3 \quad = \quad 16k_1^{(4)} \tag{30}$$

$$L_4 \quad = \quad 32k_2^{(4)}(k_2^{(4)} + k_2'^{(4)}) \tag{31}$$

$$L_5 \quad = \quad 4(3k_1^{(4)} + k_1'^{(4)}) \tag{32}$$

$$L_6 \quad = \quad 16(k_2^{(4)} + k_2'^{(4)})^2 \tag{33}$$

$$L_7 \quad = \quad 0 \tag{34}$$

$$L_8 \quad = \quad 4(k_1^{(4)} + k_1'^{(4)}) \tag{35}$$

$$H_2 \quad = \quad 8(k_1^{(4)} + k_1'^{(4)}). \tag{36}$$

Now we can see whether the relation (26)-(36) can at all fit the experimental values given in Ref[8, 10] by a suitable choise of k's. A square fitting of L's leads to the following numbers:

$$k_1^{(4)} \quad = \quad -0.24 \times 10^{-3}$$

$$k_1'^{(4)} \quad = \quad 0.78 \times 10^{-3}$$

$$k_2^{(4)} \approx k_2'^{(4)} = i1.84 \times 10^{-3} \; ; \; i = \sqrt{-1} \; . \tag{37}$$

These values can be used as fit parameters in order to obtain small deviation of L_i parameters from their experimental values (in unit 10^{-3})

$$
\begin{aligned}
L_1 &= 0.4 & (0.7 \pm 0.3) \\
L_2 &= 1.0 & (1.3 \pm 0.7) \\
L_3 &= -3.9 & (-4.4 \pm 2.5) \\
L_4 &= -0.2 & (-0.3 \pm 0.5) \\
L_5 &= 0.2 & (1.4 \pm 0.5) \\
L_6 &= -0.2 & (-0.2 \pm 0.3) \\
L_7 &= 0 & (-0.4 \pm 0.15) \\
L_8 &= 2.1 & (0.9 \pm 0.3) \; ,
\end{aligned} \tag{38}
$$

where the values in parenthesis are the experimental values[8, 10]. Also one can find some simple relations between L_i's, for instance by putting $k_2^{(4)} = k_2'^{(4)}$, we find:

$$
\begin{aligned}
L_2 &= \frac{1}{2}(L_8 - L_5) \\
L_3 &= -4L_2 \\
L_4 &= L_6 \\
L_6 &= 4L_1 - 2L_2.
\end{aligned} \tag{39}
$$

4   Anomaly

By now it seems that there is no possibility in our formalism to obtain an anomaly cancellation term, (i.e. WZW term) naturally. Besides, L_7 which is related to $U_A(1)$ anomaly[11] is also zero. To solve the problem of WZW term, we should change our geometrical space and take a 5-dimensional space. Again we take the algebra $\mathcal{A}$ as (13), but this time $\mathcal{M}$ is a 5-dimensional spin manifold. In fact we use the idea of Kaluza-Klein for the continuous part of our geometrical space. As is well known, there is no gradation matrix similar to γ_5 in the 5-dimensional space, so for Dirac operator we take it as follows:

$$D = \begin{pmatrix} \not{\partial} \otimes \mathbb{1} & \mathbb{1}' \otimes M \\ \mathbb{1}' \otimes M^\dagger & 0 \end{pmatrix} \, , \tag{40}$$

where $\not{\partial} = \Gamma_i \partial^i$ and Γ_i are the Dirac matrices for 5-dimensional space. By $\mathbb{1}'$, we mean the unit matrix in internal spin space. Now it is straightforward to calculate all the $\mathcal{L}^{(i)}$s in the same manner as we have done in the previous section, but up to 5th order of momentum. Again, we find that $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(3)}$ will vanish, but $\mathcal{L}^{(5)}$ which is taken as:

$$\mathcal{L}^{(5)} = Tr(k^{(5)} L^5) + (other \ terms \ in \ 5th \ order \ of \ momentum) \tag{41}$$

will yield only one none vanishing term which is the WZW term,

$$\mathcal{L}^{(5)} = k^{(5)} \epsilon^{ijklm} Tr(U\partial_i U^\dagger \partial_j U \partial_k U^\dagger \partial_l U \partial_m U^\dagger) \ ; \ i,j,k,l,m = 0,1,2,3,5. \tag{42}$$

This term is topological i.e. in action this term can be integrated over the 4-dimensional space which is the boundary of $\mathcal{M}$[12]. Other terms ($\mathcal{L}^{(2)}$ and $\mathcal{L}^{(4)}$) will have the same form as before except for the terms which contain $\partial_5 U$. We will omit them because, we are in the low energy regime and can not observe the 5th dimension (i.e. the same as in the Kaluza-Klein program). By this strategy what will remain is the ChPL up to order four of momentum and WZW term (except for the external field dependent terms). Unfortunately the problem of $U_A(1)$ anomaly term will remain in this approach.

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